An Introduction to Theories of Turbulence

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Topics not included

(recent papers/theses, open for discussion during this visit)

- 1. Turbulent combustion
- 2. Turbulent mixing
- 3. Inertial Confinement Fusion: UQ
- 4. Inertial Confinement Fusion: fluid transport
- 5. Short term weather forecasts of cloud cover
- 6. Cardiac electrophysiology and fibrillation
- 7. An API for Front Tracking
- 8. Financial modeling

The Navier-Stokes Equation Incompressible Fluid Dynamics (*NS*) $\partial_t v + \nabla \Box (v \otimes v) + \nabla P = \mu \Delta v$ (conservation of momentum)

 $\nabla \Box v = 0$ (incompressibility)

P is not a dynamic variable $\nabla \Box \nabla P = \Delta P = \nabla (\text{NS v terms})$ $P = \Delta^{-1} (\nabla (\text{NS v terms}))$ The Navier-Stokes Equation Incompressible Fluid Dynamics

$$(NS) \quad \partial_t v + \nabla \Box (v \otimes v) + \nabla P = \mu \Delta v$$

(conservation of momentum)

- $\nabla \Box v = 0$ (incompressibility)
- Solve for $v(t + \Delta t)$ given v(t)
- Finite difference, trial solution v^*
- Hodge decomposition,
- v = incompressible (divergence free) part of v *

Multiscale Science

- Problems which involve a span of interacting length scales
 - Easy case: fine scale theory defines coefficients and parameters used by coarse scale theory
 - Example: viscosity in Navier-Stokes equation, comes from Boltzmann equation, theory of interacting particles, or molecular dynamics, with Newton's equation for particles and forces between particles

Multiscale

- Hard case
 - Fine scale and coarse scales are coupled
 - Solution of each affects the other
 - Generally intractable for computation
 - Example:
 - Suppose a grid of 1000^3 is used for coarse scale part of the problem.
 - Suppose fine scales are 10 or 100 times smaller
 - Computational effort increases by factor of 10⁴ or 10⁸
 - Cost not feasible
 - Turbulence is classical example of multiscale science

Origin of Multiscale Science as a Concept

- author = "J. Glimm and D. H. Sharp",
- title = "Multiscale Science",
- journal = "SIAM News",
- year = "1997",
- month = Oct.

Turbulence Theories

- Many theories, many papers
- Last major unsolved problem of classical physics
- New development
 - Large scale computing
 - Computing in general allows solutions for nonlinear problems
 - Generally fails for multiscale problems

Four Useful Theories for Turbulence

- Large Eddy Simulation (LES) and Subgrid Scale Models (SGS)
- Kolmogorov 41
- Renormalization group
- PDF convergence in the LES regime

Need for turbulence models

- What is turbulence?
 - Complex flow characterized by many interacting vortices, of all different sizes
- What is a model?
 - What we do when effects are too fine scale to compute
 - Effects of unresolved scales on the resolved scales

LES and SGS

 Based on the idea that effect of small scales on the large ones can be estimated and compensated for.

Levels of accuracy in computing

- DNS = direct numerical simulation
 - All important length scale are resolved numerically.
 - Rarely practical for most important problems
- LES = large eddy simulations
 - Compute some but not all of the turbulent scales
 - Model the scales not computed
 - How to assess convergence
 - In terms of probability distributions, ie. pdfs
- RANS = Reynolds averaged Navier-Stokes
 - Model all of the turbulent length scales
 - Simulate only the largest, problem dependent ones

Derivation of Models

- Models can be postulated (and not derived), parameters fit to experiments
 - Typically the required parameters change from one flow to another, so this type of model is unsatisfactory
- Models can be derived, parameters determined numerically from the simulation
 - Second class of models are called dynamic
 - No free parameters

Subgrid models for turbulence

- Typical equations have the form
- Averaged equations:

 $U_t + \nabla F(U) = \mathcal{E} \Delta U$

 $\begin{aligned} U_t + \nabla F(U) &= \varepsilon \Delta U \\ \overline{F(U)} \neq F(\overline{U}) \\ \overline{F(U)} \approx F(\overline{U}) + F_{SGS}(\overline{U}) \end{aligned}$

 F_{SGS} is the subgrid scale model and corrects for grid errors

Subgrid models for turbulent flow

$$F_{SGS}\left(\overline{U}\right) \equiv F\left(\overline{U}\right) - \overline{F\left(U\right)}$$

$$\overline{U_{t}} + \nabla F\left(\overline{U}\right) = \varepsilon \Delta U + \nabla F_{SGS}\left(\overline{U}\right)$$

$$F_{SGS}\left(\overline{U}\right) \Box \varepsilon_{\text{turbulent}} \nabla \overline{U} \text{ (key modeling step)}$$

$$\overline{U_{t}} + \nabla F\left(\overline{U}\right) = \left(\varepsilon + \varepsilon_{\text{turbulent}}\right) \Delta \overline{U}$$

Subgrid Scale Models (Moin et al.)

- No free (adjustable) parameters in the SGS terms
- Parameters are found dynamically from the simulation itself
- After computing at level Delta x, average solution onto coarser mesh. On coarse mesh, the SGS terms are computed two ways:
 - Directly on coarse mesh as on the fine mesh, with a formula
 - Indirectly, by averaging the fine grid terms to be closed onto the coarse grid.
 - Identity of two determinations for SGS terms becomes an equation for the coefficient (on the coarse grid), otherwise missing.
 - Assume: coefficient has a known relation to Delta x and otherwise is determined by an asymptotic coefficient. Thus on a fine LES grid the coefficient is known by this reasoning.

Definitions



LES Challenge

What is the coefficient for the Smagorinsky model for turbulent diffusion?

Answer: next 4 slides (same answer, 4 times)

Dynamic Turbulent Closures

- Unknown coefficients come from simulation itself, thus they are called dynamic
- Form of averaged nonlinear expression is a gradient. Basic idea due to Smagorinsky
 - Gives rise to turbulent viscosity, turbulent thermal or mass diffusion
 - Missing and problem dependent coefficient to be determined
- Turbulent closures express the influence of small scales (not resolved numerically) on the large scales (resolved)
- Dynamic closures: one of the few successful and widely accepted theories of turbulence.

Two Filter Levels

- Mesh block average = first filter level
- Average over array of mesh blocks (2x2x2) = test filter level
- At test filter level, compute nonlinear term in two ways
- Assume nonlinear term has a fixed form ("model") times an unknown coefficient, and coefficient is a universal constant.
- Comparison of two computations yields an equation
- Common (universal) coefficient in each of the two computations is determined from this new equation

Bar = filter; hat = test filter

 $T = \text{Mesh level } R_{ij} = \text{Test level average of mesh level } R_{ij}$

$$\tau = R_{ij}(\bar{v}) = R_{ij}$$
 (Test level averaged v)

 $T, \tau =$ Computable term + unknown term

Unknown term is identical

 $L = T - \tau$ = Leonard stress = computable terms

 $T, \tau =$ (common coefficient) x (common model with scale factor)

L gives equation for coefficient

L = computable terms =

common coefficient x (common model) (difference in scale factors) common coefficient = L (common model)⁻¹(difference in scale factors)⁻¹

Bar = filter; hat = test filter

$$T_{ij} = -\overline{\rho v_i v_j} + \overline{\rho v_i} \overline{\rho v_j} / \overline{\rho}$$

$$L_{kk}^{I} = T_{kk} - \tau_{kk} = -\frac{\overline{\rho v_k}}{\overline{\rho}} \overline{\rho v_k} + \frac{\overline{\rho v_k} \overline{\rho v_k}}{\overline{\rho}}$$

$$M_{kk}^{I} = C_I 2\Delta^2 \left(-\rho \left| S \right|^2 + \overline{\rho} \left| S \right|^2 \right)$$

$$C_I = L_{kk}^{I} / M_{kk}^{I}$$

Idea of computation, in words

The key idea is two different filter levels, fine and course At the course level, the SGS can be computed in two distinct ways.

Each has 2 terms, one that can be evaluated from grid level information and one that cannot. The difference of the two thus has 4 terms.

But by a miracle, the two non-computable terms cancel. So the difference is computable from grid level information.

Set the difference equal to the difference of the models (gradient diffusion models), these are both computable, but with unknown coefficients.

Assume identical coefficients (after scaling out delta x factor). Result is one equation for one unknown coefficient.

At least in the most favorable case, isotropic L_kk case. Otherwise, many equations one coefficient, use least squares for best choice of coefficient.

Kolmogorov 41

- The opposite and also very successful idea in turbulence is that the main coupling and influence between length scales is that large scale motions (eddies) influence small scale motions (eddies) but not the opposite.
- Distinguish three ranges of length scales:
 - Large scale motions, very problem dependent and nonuniversal. Called the energy containing eddies
 - Intermediate scale motions, called the inertial range, because governed by Euler, not Navier-Stokes effects
 - Dissipation range, in which viscosity plays a role. Starts at a length scale for which the flow is laminar (nonturbulent), called the Kolmogorov scale
- K41 is a theory of the inertial range. It is a theory for E. It is a theory for E(k) as a function of k, for frequencies corresponding to the inertial range

 $\langle \rangle$ = spatial, temporal or ensemble average

$$u' = u - \langle u \rangle$$
$$E = \langle u'u' \rangle =$$

turbulent kinetic energy

K41 and E(k)

$$E(t) = \frac{1}{2} \iiint_{R^3} u^2(\bar{x}, t) d_3 \bar{x} =$$

$$\int_0^\infty 4\pi k^2 \iint \left| u^2 \right| (t, \bar{k}) d_2 \bar{k} dk = \int_0^\infty E(t, k) dk$$

$$E(t,k) = \alpha \varepsilon^{2/3} k^{-5/3}$$

G. Batchelor, The theory of homogeneous Turbulence. Cambridge University Press. 1955, Chapter 6

Dimensional Analysis

$$\operatorname{Re} = \operatorname{Re}(v, l) = ul / v$$

Re = 2,000 to 10,000 typical for transition to turbulence

$$\varepsilon = \frac{3}{2} \frac{du^2}{dt} = 2\nu \int_0^\infty k^2 E(k,t) dk$$

= rate of removal of kinetic energy due to viscosityDimensionless (Kolmogorov) length and velocity scales

$$\eta = \left(\frac{v^3}{\varepsilon}\right)^{1/4} u = (v\varepsilon)^{1/4}$$

$$[v] = [l^2 t^{-1}]; [\varepsilon] = [l^2 t^{-3}]; [v^3 \varepsilon^{-1}] = [l^4]; [v\varepsilon] = [l^4 t^{-4}]$$

$$\operatorname{Re}(v, \eta) = \frac{\eta u}{v} = \frac{v^{3/4} \varepsilon^{-1/4} (v\varepsilon)^{1/4}}{v} = 1$$

Inertial range is $\eta \ll r \ll l$

K41: Hypothesis of Universal Tubulence in the Inertial Range

• Dimension [] of E(k):

$$[E(k)] = [d^{2}k][E(\vec{k})]$$
$$[l^{-2}][l^{2} / t^{2}][l^{3}] = [l^{3} / t^{2}] = [u^{2}l]$$

- E(k) must have the identical form for every k in the inertial range, by the universal assumption.
- Thus E(k) can depend only on k and epsilon

K41, concluded

 $[E(k)] = [l^3][t^{-2}] = [\varepsilon^a][k^b] =$ $= [l^{2a}t^{-3a}][l^{-b}] = [l^{2a-b}][t^{-3a}]$ 3a = 2; a = 2/32a - b = 3; b = -5/3 $E(k) = \alpha \varepsilon^{a} k^{b} = \alpha \varepsilon^{2/3} k^{-5/3}$

The renormalization group (RNG) and scale invariant theories Inertial range turbulence is scale invariant. It has no preferred length scales.

The Navier-Stokes equation is not scale invariant. The viscosity introduces a fundamental length scale, the Kolmogorov scale $\eta = (v^3 / \varepsilon)^{1/4}$ at which viscous effects become important.

The Euler equations are scale invariant.

Change of length scale

Change of length scale gives an equation; this equation is especially simple and important for scale invariant theories.

The change of scale equation is a fundamental part of RNG. If the change is stepwise, ie $l \rightarrow l/2$ then the equation becomes a group operation, hence the term RNG.

RNG has three basic steps.

Three RNG Steps

- Apply the change of scale to integrate the model equations for one length scale, either differentially or stepwise.
- 2. Modify certain parameters in the theory, to achieve effective agreement with experiment, or otherwise avoid modifying the problem in an essential manner.
- Reformulate (rescale) the full problem, to keep the currently active scale at a fixed size.

The Result

The result, if it converges, is an RNG fixed point, that is scale invariant. For a scale invariant problem, the fixed point is often of especial significance, leading for example to scaling laws and exponents, as in K41.

The bad news: all problem specific details are removed in the rescaling step. le, the original problem -> infinite distance.

A 2 Step RNG for LES

LES is basically steps 1, 2 of the RNG.

Mesh refinement is the basic integration step, while the SGS subgrid terms are the basic parameter resetting step.

We skip the rescaling step. Then scale invariance emerges as a small length scale asymptotic property, to which the RNG power laws apply.

We have the best of both worlds, the RNG scaling laws and the important problem specific details.

AN RNG Reformulation of LES

author = "J. Glimm and W. Hu and H. Lim and B. Plohr and D. H. Sharp",

title = "Large Eddy Simulation, Turbulent Transport and the Renormalization Group",

year = "2016", journal = "Ann. Math. Sci. and Applications", volume = "1",

pages = "149-180",

note = "Los Alamos Preprint LA-UR-12-26149. Stony Brook University Preprint Number SUNYSB-AMS-15-05",

PDF Convergence

- author = "G.-Q. Chen and J. Glimm",
- title = "Kolmogorov's Theory of Turbulence and Inviscid Limit of the Navier-Stokes equations in R³",
- year = "2010",
- journal = "Commun. Math. Phys.",

Definitions

- Weak solution
 - Multiply Navier Stokes equation by test function, integrate by parts, identity must hold.
- L_p convergence: in L_p norm
- w* convergence for passive scalars chi_i
 - Chi_i = mass fraction, thus in L_\infty.
 - Multiply by an element of dual space of L_\infty
 - Resulting inner product should converge after passing to a subsequence
 - Theorem: Limit is a PDF depending on space and time, ie a measure valued function of space and time.
 - Theorem: Limit PDF is a solution of NS + passive scalars equation.